Problem Set 9 Solutions

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5.2.1

The given problem can be rewritten as

$$\text{minimize } \sum_{i=0}^{m} f_i(x_i)$$

$$\text{subject to } x_i \in X_i, \quad i = 0, 1, \ldots, m$$

$$x_i = x_0, \quad i = 1, 2, \ldots m.$$

The dual function for this problem is given by

$$q(\lambda_1, \ldots, \lambda_m) = \min_{x_i \in X_i, i=0,\ldots,m} \{\sum_{i=0}^{m} f_i(x_i) + + \sum_{i=0}^{m} \lambda_i'(x_i - x_0)\}$$

or equivalently

$$q(\lambda_1, \ldots, \lambda_m) = \min_{x \in X_0} \{f_0(x) - (\lambda_1 + \cdots + \lambda_m)'x\} + \sum_{i=1}^{m} \min_{x \in X_i} \{f_i(x) + \lambda_i' x\}$$

for $\lambda_1, \lambda_2, \ldots, \lambda_m \in R^n$.
By introducing functions

$$q_0(\lambda) = \min_{x \in X_0} \{f_0(x) - \lambda' x\},$$

$$q_i(\lambda) = \min_{x \in X_i} \{f_i(x) + \lambda' x\}, \quad i = 1, \ldots, m,$$

the dual problem reduces to

$$\text{maximize } \; q_0(\lambda_1 + \cdots + \lambda_m) + \sum_{i=1}^{m} q_i(\lambda_i)$$

$$\text{subject to } \; \lambda_i \in R^n, \quad i = 1, \ldots, m.$$

Because the primal feasible set $\cap_{i=1}^m X_i$ is nonempty and compact, and $f(x) = \sum_{i=1}^m f_i(x)$ is continuous over [since it is convex over], by Weierstrass theorem, the primal optimal solution exists. Furthermore, according to Prop. 5.2.1, there is no duality gap and the dual optimal solution exists.